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# SYMMETRIC JUMP PROCESSES[1]

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ABSTRACT. We use the theory of Dirichlet spaces to construct symmetric Markov processes of pure jump type and to identify the Lévy measures for these processes. Particular attention is paid to lattice and hard sphere systems which interact through speed change and exclusion.

**Introduction.** Our starting point is a family of measures $\pi(x, dy)$ on a separable locally compact Hausdorff space $\mathbf{X}$ and a fixed Radon measure $\mu$ on $\mathbf{X}$ which together satisfy a condition of symmetry and certain mild conditions of regularity. These generate a regular Dirichlet space to which corresponds in the sense of [3] and [13] a strong Markov process. The problem considered in this paper is that of identifying the given measures $\pi(x, dy)$ as the "Levy measures" for the Markov process. At present we are able to obtain satisfactory results only after imposing one of two side conditions labeled 1.5 and 1.6 below. These are too restrictive for a general theory but permit us to treat some interesting examples.

The main results are established in §1 and the appendix. §§2, 3 and 4 are devoted to examples.

Symmetric stable processes on Euclidean space are considered in §2. These are included both to indicate the scope of one of our side conditions and also in preparation for our treatment of hard sphere systems in §4. The results themselves are of little interest since they follow easily from the standard representation of such processes as integrals of differential Poisson processes.

A class of interacting lattices introduced by F. Spitzer are considered in §3. Our approach provides a construction of these processes which is quite different from the one in [6] and [7]. However the latter results are more complete in that the relevant semigroup is shown to map continuous functions into continuous functions. This means in particular that any initial configuration is permissible and that certain exceptional polar sets which are always present with our approach can be eliminated.

Systems of interacting hard spheres are treated in §4. Centers of the individual spheres move according to a fixed symmetric stable process with index $\alpha < 1$. The interaction is a combination of speed change and hard sphere exclusion. Our existence theorem here is new and does not seem to be obtainable by the more

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familiar semigroup techniques. The restriction to $\alpha < 1$ is essential for our present tools, but we doubt that the case $1 \leq \alpha < 2$ is intrinsically different.

Throughout the paper $\mathbf{X}$ is a separable locally compact Hausdorff space and $C_{\text{com}}(\mathbf{X})$ is the collection of continuous functions on $\mathbf{X}$ having compact support. All functions are real valued. The symbols $I(A)$ and $1_A$ will both be used for the indicator of the set $A$ and the symbol $E(A; B; f)$ will be used for the integral of $f$ over the set $A \cap B$. Notations and results in §§1 through 3 of [13] will often be used without explicit reference.

1. **General theory.** We consider a "generalized matrix on $\mathbf{X}$." This is a family $\pi(x, dy)$ of Borel measures on $\mathbf{X}$ indexed by $x$ in $\mathbf{X}$ and satisfying

1.1.1. $\pi(x, \{x\}) = 0$ and $\pi(x, \cdot)$ is a Radon measure on $\mathbf{X} - \{x\}$.

1.1.2. $\pi(\cdot, \Gamma)$ is Borel measurable on $\mathbf{X}$ for every Borel subset $\Gamma$ of $\mathbf{X}$.

1.1.3. $\pi(\cdot, \mathbf{X} - D)$ is locally bounded on any open set $D$.

**Lemma 1.1.** *Let $\mu$ be a Radon measure on X. Then there is a unique sigma finite Borel measure $\pi[\mu(dx), dy]$ on $\mathbf{X} \times \mathbf{X}$ such that*

$$\iint \pi[\mu(dx), dy] F(x,y) = \int \mu(dx) \int \pi(x, dy) F(x,y)$$

*whenever $F(x,y)$ is nonnegative and jointly Borel measurable on* $\mathbf{X} \times \mathbf{X}$.

**Proof.** The existence of $\pi[\mu(dx), dy]$ depends on Borel measurability of $f(x) = \int \pi(x, dy) F(x,y)$. This can be shown by standard arguments when $F$ is bounded and supported on a product set $K \times L$ with $K$ and $L$ disjoint compact sets and then the general case follows by an obvious splitting and passage to the limit. Sigma finiteness is immediate since the diagonal will have measure zero.

1.1. **Definition.** *A Radon measure $\mu$ is symmetric if the measure $\pi[\mu(dx), dy]$ is symmetric on* $\mathbf{X} \times \mathbf{X}$.

**Lemma 1.2.** *Let $\mu$ be symmetric. If $f(x) = f'(x)$ [a.e. $\mu(dx)$] on* $\mathbf{X}$, *then also $f(x) - f(y) = f'(x) - f'(y)$ [a.e. $\pi[\mu(dx), dy]$] on* $\mathbf{X} \times \mathbf{X}$.

**Proof.** It suffices to observe that $\Gamma \times \mathbf{X}$ and therefore $\mathbf{X} \times \Gamma$ is a $\pi(\mu, \cdot)$ null subset of $\mathbf{X} \times \mathbf{X}$ whenever $\Gamma$ is a $\mu$ null subset of $\mathbf{X}$.

1.2. **Notation.** Let $\mu$ be symmetric. Then $\pi_\mu\langle f, g\rangle$ is defined by

$$\pi_\mu\langle f, g\rangle = \iint \pi[\mu(dx), dy]\{f(x) - f(y)\}\{g(x) - g(y)\}$$

when it converges. $\mathbf{L}_\mu$ is the collection of $f$ in $L^2(\mu)$ such that $\pi_\mu\langle f, f\rangle$ is finite.

It is easy to check that $(\mathbf{L}_\mu, \frac{1}{2}\pi_\mu)$ is a Dirichlet space relative to $L^2(\mu)$ and in particular the linear space $\mathbf{L}_\mu$ is a Hilbert space relative to any of the inner products

$$\pi_{\mu,u}\langle f, g\rangle = \pi_\mu\langle f, g\rangle + u \int \mu(dx) f(x) g(x)$$

defined for $u > 0$. This Dirichlet space always determines a submarkovian resolvent on $L^2(\mu)$ (see [4, §2]) but in general this resolvent need not correspond to a Markov process with "well behaved" sample paths. In order to obtain such a process we impose a regularity condition and replace $\mathbf{L}_\mu$ by a closed subspace. Thus we consider only symmetric $\mu$ satisfying the following

1.3. **Regularity condition.** $\mathbf{L}_\mu \cap C_{\text{com}}(\mathbf{X})$ is uniformly dense in $C_{\text{com}}(\mathbf{X})$.

This regularity condition does not seem to be a serious restriction in practice.

1.4. **Definition.** *The jump Dirichlet space generated by the symmetric measure* $\mu$ *is the pair* $(\mathbf{F}_\mu, E_\mu)$ *where* $E_\mu = \frac{1}{2}\pi_\mu$ *and where* $\mathbf{F}_\mu$ *is the closure in* $\mathbf{L}_\mu$ *of* $\mathbf{L}_\mu \cap C_{\text{com}}(\mathbf{X})$. *That is,* $\mathbf{F}_\mu$ *is the collection of* $f$ *in* $\mathbf{L}_\mu$ *for which there exists a sequence* $f_n$, $n \geq 1$ *in* $\mathbf{L}_\mu \cap C_{\text{com}}(\mathbf{X})$ *such that*

$$\int \mu(dx)\,|f(x) - f_n(x)|^2 \to 0, \qquad \pi_\mu\langle f - f_n, f - f_n\rangle \to 0,$$

*as* $n \uparrow \infty$.

**Remark.** In checking that $(\mathbf{F}_\mu, E_\mu)$ is indeed a Dirichlet space, the only nontrivial property is contractivity. But if $f$ in $\mathbf{F}_\mu$ can be approximated by the sequence of functions $f_n$ in $\mathbf{L}_\mu \cap C_{\text{com}}(\mathbf{X})$ and if $T$ is a normalized contraction, then the Dirichlet norms of $Tf_n$ are bounded and so $Tf$ can be approximated by Cesàro sums of a subsequence from the $Tf_n$. (See [11, p. 80].)

If $\mu$ is a symmetric measure satisfying the regularity condition 1.3, then $\mathbf{F}_\mu \cap C_{\text{com}}(\mathbf{X})$ is both uniformly dense in $C_{\text{com}}(\mathbf{X})$ and dense in $\mathbf{F}_\mu$ relative to any of the inner products $\pi_{\mu,u}$. Thus, except for the irrelevant condition that $\mu$ be dense, the pair $(\mathbf{F}_\mu, E_\mu)$ is a regular Dirichlet space in the sense of [13]. So by the results of §2 in that paper there exists an exceptional set $N$ of $\mathbf{X}$ which is "negligible" in an appropriate sense and a family of probabilities $\mathcal{P}_x$ indexed by $x$ in the complement $\mathbf{X} - N$ which form a strong Markov process taking values in $\mathbf{X} - N$ and having the usual regularity conditions. That is, the process is quasi-left-continuous and the trajectories are right continuous with left-hand limits existing everywhere. We would like to show that the measures $\pi(x, dy)$ are in the sense of [16] the Lévy measures for this Markov process. At present we can do this only after imposing one of two side conditions.

In the remainder of this section we will be using freely the notation of [13, §3]. For example the operators $G_u^D$, $u > 0$, are the resolvent operators which correspond to the process killed upon exiting from the set $D$ and $(\mathbf{F}^D, E^D)$ is the corresponding Dirichlet space. Here and elsewhere the subscript $\mu$ will usually be suppressed.

1.5. **Side condition.** There exists a subbasis of open sets $D$ such that the indicator $1_D$ belongs to the killed Dirichlet space $\mathbf{F}^D$.

1.6. **Side condition.** $\mathbf{F} = \mathbf{L}$ and there exists a collection of open sets which is closed under finite unions and which is a basis for the topology of $\mathbf{X}$ such that every $D$ in this collection has the following property: for quasi-every $x$ in $D$

$$(1.1) \qquad \inf\{t > 0\colon X_t \text{ is not in } D\} = \inf\{t > 0\colon X_t \text{ is not in } \mathrm{cl}(D)\}$$

with $\mathcal{P}_x$ probability one. ($\mathrm{cl}(D)$ denotes the closure of $D$.)

**Remark.** In any examples where we have been able to establish the first part of 1.6 we have found the second part to be trivial. Therefore we have not explored the possibility that the second part is actually superfluous.

The point of these side conditions is

**Theorem 1.3.** *If either* 1.5 *or* 1.6 *is satisfied, then there exists a collection of open sets which is closed under finite unions and which is a basis for the topology of* $\mathbf{X}$ *such that for each $D$ in the collection and for each $u > 0$*

$$G_u^D\{u1 + \pi(\cdot, \mathbf{X} - D)\} = 1 \tag{1.2}$$

*quasi-everywhere on $D$.*

Equation (1.2) effectively identifies $\pi(\cdot, \mathbf{X} - D)$ as the density for "jumping out of $D$." We prove now that 1.5 is sufficient for (1.2). The proof that 1.6 is sufficient is rather technical and we defer it until the appendix.

Fix an open set $D$ satisfying 1.5. By [13, Theorem 3.6] the function space $\mathbf{F}^D$ is the collection of $f$ in $\mathbf{F}$ such that the quasi-continuous refinement $f^* = 0$ quasi-everywhere on $M = \mathbf{X} - D$ and

$$E^D(f,g) = \tfrac{1}{2}\pi_\mu^D\langle f,g\rangle + \int_D \mu(dx)\pi(x, M)f(x)g(x)$$

for $f$, $g$ in $\mathbf{F}^D$. Here

$$\pi_\mu^D\langle f,g\rangle = \iint_{D\times D} \pi[\mu(dx), dy]\{f(x) - f(y)\}\{g(x) - g(y)\}.$$

Since the indicator $1_D$ belongs to $\mathbf{F}^D$ we have for $f$ in $\mathbf{F}^D$

$$E^D(1_D, f) = \int \mu(dx)\pi(x, M)f(x)$$

from which (1.2) follows by Proposition 1.2 and (1.3) in [13]. Thus Theorem 1.3 will be proved with the hypothesis 1.5 if we show that the intersection $D = D_1 \cap D_2$ and the union $D' = D_1 \cup D_2$ satisfy 1.5 whenever $D_1$ and $D_2$ do. By the contraction property $\mathbf{F}$ is closed under products of bounded functions and so the indicators $1_D = 1_{D_1}1_{D_2}$ and $1_{D'} = 1_{D_1} + 1_{D_2} - 1_{D_1}1_{D_2}$ belong to $\mathbf{F}$. Moreover the quasi-continuous refinements $1_{D_1}^*$ and $1_{D_2}^*$ vanish quasi-everywhere on $M_1$ and $M_2$ and so $1_{D'}^*$ vanishes quasi-everywhere on $M'$. Thus $1_{D'}$ belongs to $\mathbf{F}^{D'}$. A similar argument establishes this result for $D$.

*In the remainder of this section we assume that the conclusion of Theorem* 1.3 *is valid.*

Fix an open subset $D$ of $\mathbf{X}$ satisfying (1.2) and let $M = \mathbf{X} - D$ as above. It follows that if $D_n$, $n \geq 1$, is an increasing sequence of open sets each having compact closure in $D$ and such that $D_n \uparrow D$, then for $u > 0$ $H_u^{\mathbf{X}-D_n}1_D \downarrow 0$ quasi-everywhere on $D$ as $n \uparrow \infty$ and therefore

$$\mathcal{P}_x[\sigma(M \cup \{\partial\}) < \infty; X_{\sigma(M\cup\{\partial\})-0} \text{ is in } M] = 0.$$

That is, the trajectory can exit from $D$ only by "jumping out" and never by "wandering to the boundary of $D$." The proof of Theorem 1.1 in [14] suffices to refine (1.2) to

$$\mathcal{E}_x[\sigma(M \cup \{\partial\}) < R_u; f(X_{\sigma(M\cup\{\partial\})-0})] = G_u^D(f\pi(\cdot, M))(x) \tag{1.3}$$

for $f \geq 0$ on $D$ and for quasi-every $x$ in $D$. Next let $D'$ be a second open set containing $D$ and satisfying (1.2). Then

$$\mathcal{P}_x[\sigma(M) < R_u; X_{\sigma(M)} \text{ is in } D'] = G_u^D \pi(\cdot, D' \cap M)(x) \tag{1.4}$$

for quasi-every $x$ in $D$. This follows since the left side

$$\begin{aligned}
&= \mathcal{E}_x[\sigma(M) < R_u; 1_{D'}(X_{\sigma(M)})] \\
&= \mathcal{E}_x[\sigma(M) < R_u; G_u^{D'}\{u1 + \pi(\cdot, M')\}(X_{\sigma(M)})] \\
&= H_u^M G_u^{D'}\{u1 + \pi(\cdot, M')\}(x) \\
&= G_u^{D'}\{u1 + \pi(\cdot, M')\}(x) - G_u^D\{u1 + \pi(\cdot, M')\}(x) \\
&= G_u^D\{u1 + \pi(\cdot, M)\}(x) - G_u^D\{u1 + \pi(\cdot, M')\}(x) \\
&= G_u^D \pi(\cdot, D' - D)(x).
\end{aligned}$$

Again the proof of Theorem 1.1 in [14] refines (1.4) to

$$\mathcal{E}_x[\sigma(M) < R_u; f(X_{\sigma(M)-0}); X_{\sigma(M)} \text{ is in } D'] = G_u^D f\pi(\cdot, D' - D)(x). \tag{1.5}$$

and then an easy passage to the limit extends (1.5) to

$$\begin{aligned}
&\mathcal{E}_x[\sigma(M) < R_u; f(X_{\sigma(M)-0})g(X_{\sigma(M)})] \\
&\qquad = \mathcal{E}_x \int_0^{\sigma(M)} dt e^{-ut} f(X_t) \int_M \pi(X_t, dy) g(y)
\end{aligned} \tag{1.6}$$

valid for $f \geq 0$ on $D$ and $g \geq 0$ on $M$. This effectively identifies the measures $\pi(x, dy)$ as the Lévy measures for the process. We finish this section by establishing the formula (1.10) below for the Dirichlet norm which gives precise meaning to the heuristic statement: "the process consists entirely of jumps."

For $u > 0$ let $\mathcal{P}^u$ be the countably additive measure constructed in [14, §4]. Define the martingale $M_u f(t)$ by

$$M_u f(t) = f(X_t) + \int_{\zeta^*}^t ds\, g(X_s)$$

for $f = G_u g$ with $g$ bounded and $\mu$-integrable and by passage to the limit for general $f$ in **F**. (See [14, §5].) Then by the very definition of $M_u f(t)$

$$E_u(f, f) = \tfrac{1}{2}\mathcal{E}^u\{M_u f(\zeta)\}^2. \tag{1.7}$$

Let $D_1, \ldots, D_N$ be open subsets of $\mathbf{X}$ having disjoint compact closures and each satisfying (1.2). For each $i$ choose open $D_i'$ with compact closure contained in $D_i$ and let $D = D_1 \cup \cdots \cup D_N$ and $D' = D_1' \cup \cdots \cup D_N'$. Define stopping times $a(n)$ and $b(n)$ as follows.

$$a(1) = \inf\{t > 0\colon X_t \text{ is in } D'\},$$

$$b(1) = \inf\{t > 0\colon X_t \text{ is in } \mathbf{X} - D_i \text{ with } i \text{ chosen so that } \mathbf{X}_{a(1)} \text{ is in } D_i'\},$$

$$a(2) = \inf\{t > b(1)\colon X_t \text{ is in } D'\}, \text{ etc.},$$

with the understanding that $a(n)$ or $b(n) = +\infty$ if not otherwise defined. The strong Markov property together with (1.6) establishes for $f$ in $\mathbf{F}$

$$\mathcal{E}_x \sum_n I[b(n) < +\infty]\{f(X_{b(n)}) - f(X_{b(n)-0})\}^2 \tag{1.8}$$
$$= \mathcal{E}_x \sum_n \int_{a(n)}^{b(n)} dt\, e^{-ut} \int_{M[a(n)]} \pi(X_t, dy)\{f(X_t) - f(y)\}^2$$

where $M[a(n)]$ is the unique $M_i = \mathbf{X} - D_i$ such that $X_{a(n)}$ is in $D_i'$. If follows from (1.8) that

$$\mathcal{E}^u \sum \{f(X_t) - f(X_{t-0})\}^2 \tag{1.9}$$
$$\geq \sum_i \int_{D_i'} dx \int_{\mathbf{X}-D_i} \pi(x, dy)\{f(x) - f(y)\}^2,$$

and after varying $D_i$ and $D_i'$ in an appropriate manner it follows from (1.9) that

$$\mathcal{E}^u \sum \{f(X_t) - f(X_{t-0})\}^2 \geq 2E_u(f,f).$$

But $f(X_t) - M_u f(t)$ is a continuous process [14, Theorem 5.2] and so it follows from Meyer's decomposition of a square integrable martingale into its continuous and discontinuous parts [10] that

$$\mathcal{E}^u \sum \{f(X_t) - f(X_{t-0})\}^2 \leq \mathcal{E}^u\{Mf(\zeta)\}^2.$$

This together with (1.7) establishes

$$\mathcal{E}_u(f,f) = \tfrac{1}{2}\mathcal{E}^u \sum \{f(X_t) - f(X_{t-0})\}^2 \tag{1.10}$$

for $u > 0$ and $f$ in $\mathbf{F}$.

**2. Symmetric stable processes in Euclidean space.** In this section $X$ is Euclidean space $R^d$, $d \geq 1$, and $dj$ is standard Lebesgue measure. For $0 < \alpha < 2$ consider the generalized matrix

$$\pi^\circ(i, dj) = C_\alpha |i - j|^{-d-\alpha} dj \tag{2.1}$$

with $C_\alpha$ the constant

$$C_\alpha = 2^{\alpha/2}\Gamma(\alpha/2 + d/2)\{(\alpha/2)\Gamma(1 - \alpha/2)\pi^{d/2}\}^{-1}.$$

Symmetry for $dj$ follows from symmetry of $|i - j|$. The regularity condition 1.3 follows from the identity

$$\tfrac{1}{2}\pi^{\circ}_{dj}\langle f,f\rangle = \int d\lambda |f^{\sim}(\lambda)|^2 |\lambda|^{\alpha} 2^{-\alpha/2}$$

valid for $f$ in $\mathbf{L}$ with $f^{\sim}$ defined after the usual passage to the limit by

$$f^{\sim}(\lambda) = (2\pi)^{-d} \int dj e^{-i\lambda \cdot j} f(j).$$

The side condition 1.6 is valid for the entire range $0 < \alpha < 2$. The second part follows for $D$ convex by an obvious symmetry argument. To prove the first part approximate $f$ in $\mathbf{L}$ by convolutions $g = \varphi * f$ which are continuous and vanish at infinity and then approximate such $g$ by $g - g_n$ with $g_n$ an appropriate truncation of $g$. (As discussed in the remark following Definition 1.4, it suffices that the approximating sequence converge almost everywhere or in measure and have bounded Dirichlet norms.)

Side condition 1.6 is sufficient for us to be able to apply the results in §1. However it will turn out that side condition 1.5 is essential in §4. Thus it is significant that 1.5 is valid, at least for sets $D$ having "reasonable boundaries," only for the range $0 < \alpha < 1$. To see why, consider a bounded open set $D$ having a smooth boundary. (A Lipschitz condition of order one would suffice.) The calculation

$$\int_{\varepsilon}^{\infty} dt\, t^{d-1} t^{-d-\alpha} = \frac{1}{\alpha}\varepsilon^{-\alpha}$$

shows that for $j$ in $D$

$$(1/c)\{\text{dist}(j, M)\}^{-\alpha} \leq \pi^{\circ}(j, M) \leq c\{\text{dist}(j, M)\}^{-\alpha}$$

where as usual $M = R^d - D$. The constant $c > 0$ depends only on $\alpha$, $d$ and the geometry of $D$. It follows that at least for such $D$

$$\int_D dj\, \pi^{\circ}(j, M) < +\infty \tag{2.2}$$

if and only if $0 < \alpha < 1$. But (2.2) is necessary and sufficient for the indicator $1_D$ to belong to $\mathbf{L}$ and therefore $\mathbf{F}$. We restrict attention now to the range $0 < \alpha < 1$ and collect some results which will be needed in §4.

**Lemma 2.1.** *Let* $0 < \alpha < 1$.

(i) *If $B$ is an open ball, then there exists a sequence of functions $\varphi_n$ in $C_{\text{com}}(R^d)$ with supports contained in $B$ and satisfying*

$$\begin{gathered} 0 \leq \varphi_n \leq 1; \qquad \varphi_n \to 1_D[a.e.\, dj]; \\ \sup_n \pi^{\circ}_{dj}\langle \varphi_n, \varphi_n\rangle < +\infty. \end{gathered} \tag{2.3}$$

(ii) *If $G$ is a bounded open set with smooth boundary, then there exists a sequence of functions $\varphi_n$ in $C_{\text{com}}(R^d)$ all identically one on $G$ and satisfying* (2.3).

The lemma guarantees in particular that $1_B$ belongs to $\mathbf{F}^D$ when $B$ is an open ball and so side condition 1.5 is satisfied. The proof is elementary once (2.2) is established. For (i), consider smaller concentric spheres and convolve with approximations to the identity. For (ii) first convolve with approximations to the identity and then truncate and normalize.

3. **Interacting lattices.** $\mathbf{X}$ is the collection of $\{0, 1\}$ valued functions on the $d$-dimensional lattice $Z^d$. Typical elements in $Z^d$ will be denoted by $i, j, k$ and typical elements in $\mathbf{X}$ will be denoted $x, y, z$. We give $\mathbf{X}$ the usual product topology so that it is a separable compact Hausdorff space.

The interaction potential is a real valued function $V$ defined on $Z^d \times Z^d$ and satisfying

3.1.1. $V(i,i) = 0$ and $V(i,j) = V(j,i)$.

3.1.2. $\sup_i \sum_j |V(i,j)| < +\infty$.

The energy localized at the sites $i$ and $j$ is

$$U_{ij}(x) = \sum_k \{x(i)x(k)V(i,k) + x(j)x(k)V(j,k)\}.$$

The *transformation* $\sigma_{ij}$ permutes the $i$th and $j$th coordinates of $x$ in $\mathbf{X}$. Thus

$$\sigma_{ij}x(j) = x(i); \qquad \sigma_{ij}x(i) = x(j); \qquad \sigma_{ij}x(k) = x(k) \quad \text{for } k \neq i, j.$$

The *single particle matrix* is an irreducible symmetric stochastic matrix on $Z^d$ such that $\pi^\circ(i,i) = 0$ for all $i$. Thus

$$\pi^\circ(i,j) = \pi^\circ(j,i); \qquad \sum_k \pi^\circ(i,k) = \sum_k \pi^\circ(k,i) = 1.$$

For $x$ in $\mathbf{X}$ let $\pi(x, dy)$ be the unique Borel measure on $\mathbf{X}$ such that for Borel $f \geq 0$ on $\mathbf{X}$,

$$\int \pi(x,dy)f(y) = \sum \pi^\circ(i,j)\exp\{U_{ij}(x)\}f(\sigma_{ij}x)$$

with the sum taken over *unordered pairs* $i, j$ such that $\sigma_{ij}x \neq x$. It is easy to check that the measures $\pi(x, dy)$ form a generalized matrix on $\mathbf{X}$ in the sense of 1.1.

The function $\pi(\cdot, \mathbf{X} - D)$ is bounded whenever $D$ depends on only finitely many coordinates and so for any symmetric probability,

$$\tfrac{1}{2}\pi_\mu\langle 1_D, 1_D\rangle = \int_D \mu(dx)\pi(x, \mathbf{X} - D)$$

is finite and $1_D$, since it is continuous, belongs to the killed Dirichlet space $\mathbf{F}^D$. (This follows from Theorem 3.6(iii) and Lemma 1.15 in [13].) Thus the regularity condition 1.3 and the side condition 1.5 are always satisfied and the results of §1 are always applicable to symmetric probabilities. The following theorem gives a direct characterization of symmetric probabilities.

**Theorem 3.1.** *A probability $\mu$ on* $\mathbf{X}$ *is symmetric if and only if*

$$\int \mu(dx)\exp\{U_{ij}(x)\}f(\sigma_{ij}x) = \int \mu(dx)\exp\{U_{ij}(x)\}f(x) \tag{3.1}$$

*for every pair $i, j$ in $Z^d$ and for $f$ in $C(\mathbf{X})$.*

**Proof.** Sufficiency of (3.1) follows directly from the definition of the measure $\pi(x, dy)$. To establish necessity fix a pair $i, j$ in $Z^d$ with $\pi^\circ(i,j) > 0$, let $S$ be a finite subset of $Z^d$ containing $i$ and $j$, let $A$ be an atom in the sigma-algebra $\mathcal{F}_S$ generated by the $x(k)$ as $k$ runs over $S$ and apply symmetry with $f$ and $g$ the indicators of $A$ and $\sigma_{ij}A$ respectively. This yields for a given symmetric measure $\mu$,

$$\pi^\circ(i,j)\int_A \mu(dx)\exp\{U_{ij}(x)\} = \pi^\circ(i,j)\int_{\sigma_{ij}A} \mu(dx)\exp\{U_{ij}(x)\}$$

and (3.1) follows for $f$ measurable with respect to $\mathcal{F}_S$ and therefore for all $f$ in $C(\mathbf{X})$. Finally the restriction $\pi^\circ(i,j) > 0$ can be removed since $\pi^\circ$ is irreducible.

The case $V \equiv 0$ corresponds to pure exclusion in [15]. Thus for pure exclusion a probability $\mu$ is symmetric if and only if it is exchangeable in the classical sense. A well-known theorem of Hewitt and Savage states that the extremal exchangeable probabilities are precisely the product exchangeable probabilities. (See [9, Chapter VIII] for a simple proof using martingale theory.) For nonzero $V$ the role of product exchangeable probabilities seems to be played by the so-called "Gibbs distributions" which to the author's knowledge were first studied on a rigorous basis by Dobrushin [1]. In particular Dobrushin shows that there is at least one Gibbs distribution for every choice of the chemical potential. In general it is not unique. Since every Gibbs distribution satisfies (3.1) it is symmetric. Thus the results of §1 can be applied to establish the existence of Markov processes which are models for systems of particles interacting through speed change and exclusion as described heuristically in [15]. We have already mentioned in the introduction that such Markov processes have also been constructed by Holley and Liggett and that their results are more complete. In the remainder of this section we describe the connection between their construction and ours.

Let $C(\mathbf{X})$ be the collection of continuous functions on $\mathbf{X}$ and let $C_0(\mathbf{X})$ be the subcollection of $f$ in $C(\mathbf{X})$ depending only on finitely many coordinates. Consider the operator $\Omega_0$ defined on $f$ in $C_0(\mathbf{X})$ by

$$\Omega_0 f(x) = \int \pi(x, dy)\{f(y) - f(x)\}.$$

Clearly $\Omega_0$ maps $C_0(\mathbf{X})$ into $C(\mathbf{X})$. Holley [6] for a special case and Liggett [7] in general have shown that the closure $\Omega$ of $\Omega_0$ (considered as operators on the Banach space $C(\mathbf{X})$) generates a strongly continuous Markovian semigroup $\{P^t, t > 0\}$ on $C(\mathbf{X})$. The connection between this semigroup and the Dirichlet spaces $(\mathbf{F}_\mu, E_\mu)$ is described in

**Theorem 3.2.** *The following conditions are equivalent for $\mu$ a probability on* $\mathbf{X}$.

(i) *$\mu$ is symmetric in the sense of* 1.2.

(ii) $\int \mu(dx)\Omega_0 f(x)g(x) = \int \mu(dx)f(x)\Omega_0 g(x)$ *for $f$, $g$ in* $C_0(\mathbf{X})$.

(iii) $\int \mu(dx)P^t f(x)g(x) = \int \mu(dx)f(x)P^t g(x)$ *for $f$, $g$ in* $C(\mathbf{X})$ *and for* $t > 0$.

*If these conditions are satisfied, then the operators $P^t$ have unique extensions to operators $P_\mu^t$ on $L^2(\mu)$ which form a strongly continuous symmetric Markovian semigroup on $L^2(\mu)$. The generator of this semigroup is the closure $\Omega_\mu$ of $\Omega_0$ (considered as an operator on $L^2(\mu)$). The function space $\mathbf{F}_\mu$ is identical with $\mathbf{L}_\mu$ and is precisely the domain of the unique positive definite square root of the selfadjoint operator $-\Omega_\mu$. Moreover*

$$E_\mu(f,g) = \int \mu(dx)(-\Omega_\mu)^{1/2}f(x)(-\Omega_\mu)^{1/2}g(x) \quad \textit{for } f, g \textit{ in } \mathbf{F}_\mu. \tag{3.2}$$

**Proof.** Equivalence of (i) and (ii) is immediate and equivalence of (ii) and (iii) follows because the generator $\Omega$ is the closure of $\Omega_0$. The remainder of the theorem follows by elementary arguments since the closure $\Omega_\mu$ of $\Omega_0$ (considered as an operator on $L^2(\mu)$) contains the operator $\Omega$.

**Remark.** Theorem 3.2 guarantees in particular that the side condition 1.6 is always satisfied. This is because $\mathbf{X}$ has a basis of sets both open and closed and so the second part of 1.6 is automatic.

It follows from Theorem 3.2 that every symmetric probability $\mu$ is invariant for the operators $P^t$. That is $\int \mu(dx)f(x) = \int \mu(dx)P^t f(x)$ for $t > 0$ and for $f$ in $C(\mathbf{X})$. In general there are invariant probabilities for the $P^t$ which are not symmetric. (See [8].) However

**Theorem 3.3.** *Let $\mu$ be a symmetric probability which has no nontrivial representation*

$$\mu = \theta\nu + (1-\theta)\lambda \tag{3.3}$$

*with $0 < \theta < 1$ and with $\nu$, $\lambda$ symmetric probabilities. Then $\mu$ has no nontrivial representation* (3.3) *with $\nu$, $\lambda$ invariant for the $P^t$.*

**Proof.** If $\mu$ has a nontrivial representation (3.3) with $\nu$, $\lambda$ invariant for the $P^t$, then there exists a nonconstant function $\varphi$ in $L^2(\mu)$ such that $P^t\varphi = \varphi$ for $t > 0$. But then $\varphi$ is in the domain of the $L^2(\mu)$ generator $\Omega_\mu$ and $\Omega_\mu\varphi = 0$, which certainly implies that $\varphi$ belongs to $\mathbf{F}_\mu$ and $\pi_\mu\langle\varphi,\varphi\rangle = 0$. Choose $\alpha$ real such that the sets $\{x\colon \varphi(x) \geq \alpha\}$ and $\{x\colon \varphi(x) < \alpha\}$ both have positive $\mu$ probability. Then the restrictions of $\mu$ to these two sets, after appropriate normalization, yield a nontrivial representation (3.3) with $\nu$, $\lambda$ symmetric.

It follows from Theorem 3.3 that every product exchangeable probability is an extremal invariant probability for the operators $P^t$ in the case of pure exclusion. This and much more have been proved by Liggett and Spitzer using quite different techniques. For the transient case see [8]. The results for $\pi^\circ$ recurrent are more recent and have not yet appeared in finished form.

**4. Interacting hard sphere systems.** In this section $\mathbf{X}$ is the collection of $\{0, 1\}$ valued functions $x$ defined on $d$-dimensional Euclidean space $R^d$ and satisfying the restriction

$$x(i)x(j) = 0 \tag{4.1}$$

whenever $0 < |i - j| < 2r$. Here $r$ is a fixed positive number which can be interpreted as the radius of a typical hard sphere.

For $\varphi \geq 0$ on $R^d$ the enumerator $\mathfrak{S}\varphi$ is defined on $\mathbf{X}$ by

$$\mathfrak{S}\varphi(x) = \sum_i x(i)\varphi(i).$$

It is easy to check that $\mathbf{X}$ is a separable compact Hausdorff space with the topology generated by the enumerators $\mathfrak{S}\varphi$ as $\varphi$ runs over nonnegative continuous functions having compact support in $R^d$.

The interaction potential is a real valued Borel function $V$ defined on $R^d$ and satisfying

4.1.1. $V(i) = V(-i)$ and $V$ is bounded from below.

4.2.2. $V(0) = 0$ and $V(i) = +\infty$ for $0 < |i| < 2r$.

4.2.3. $|V(i)| \leq v(|i|)$ for $|i| \geq 2r$ where $v$ is a monotonically decreasing function satisfying $\int_{2r}^{\infty} dt\, t^{d-1} v(t) < +\infty$.

The *single particle matrix* $\pi^\circ(i, dj)$ is defined by (2.1) with the restriction $0 < \alpha < 1$. Of course other choices for $\pi^\circ(i, dj)$ will work as well, but the restriction on $\alpha$ is essential as regards symmetric stable processes. The *generalized matrix* $\pi(x, dy)$ is defined by

$$\int \pi(x, dy) f(y) = \sum_i x(i) \int \pi^\circ(i, dj) I(x - \varepsilon_i + \varepsilon_j \text{ is in } \mathbf{X}) \cdot \exp\left\{\sum_k x(k)V(i - k)\right\} f(x - \varepsilon_i + \varepsilon_j).$$

Here $\varepsilon_i$ is the element in $\mathbf{X}$ which is one at $i$ in $R^d$ and identically zero otherwise. Of course the indicator is inserted to suppress jumps which would violate the hard sphere constraint.

Our candidate for a symmetric probability is a Gibbs probability as defined by Dobrushin in [2]. For the reader's convenience we give with slight modification Dobrushin's proof of existence.

Fix a mean density $\rho > 0$. This plays a role similar to the chemical potential in statistical mechanics. For $S$ a bounded open subset of $R^d$ let $\mathbf{X}_S$ be the collection of bounded integral valued functions $\alpha$ on the closure $\mathrm{cl}(S)$ such that $|\alpha| = \sum \alpha(i)$ is finite. Clearly $\mathbf{X}_S$ is a separable locally compact Hausdorff space in the topology generated by the enumerators $\mathfrak{S}\varphi$ as $\varphi$ runs over nonnegative continuous functions on $\mathrm{cl}(S)$. Let $\mathbf{X}_S^*$ be the collection of $x$ in $\mathbf{X}$ such that $x(i) = 0$ for $i$ in $S$. For $x$ in $\mathbf{X}$ denote by $x_S$ and $x_S^*$ the obvious members of $\mathbf{X}_S$ and $\mathbf{X}_S^*$. Let $\lambda_S(d\alpha)$ be the usual Poisson measure over $\mathrm{cl}(S)$ with mean $\rho dj$. That is, $\lambda_S$ is the probability on $\mathbf{X}_S$ determined by the conditions

4.2.1. $\mathcal{S}\varphi_1, \ldots, \mathcal{S}\varphi_n$ are mutually independent whenever $\varphi_1, \ldots, \varphi_n$ are indicators of disjoint Borel subsets of $\mathrm{cl}(S)$.

4.2.2. $\mathcal{S}\varphi$ is a Poisson variable with mean $\rho|A|$ whenever $\varphi$ is the indicator of a Borel subset $A$ of $\mathrm{cl}(S)$. (Of course $|A|$ is the Lebesgue measure of $A$.)

For $\alpha$ in $\mathbf{X}_S$ and $\beta$ in $\mathbf{X}_S^*$ define

$$U_S(\alpha, \beta) = \tfrac{1}{2} \sum \sum \alpha(i)\alpha(j)V(i-j) + \sum \sum \alpha(i)\beta(k)V(i-k)$$

with the sums taken over $i, j$ in $\mathrm{cl}(S)$ and $k$ in the complement $R^d - S$ and then define

$$Z_S(\beta) = \int_{\mathbf{X}_S} \lambda_S(d\alpha)\exp\{-U_S(\alpha,\beta)\}.$$

For $f$ bounded and measurable on $\mathbf{X}_S$ the function

$$g(x) = \{Z_S(x_S^*)\}^{-1} \int \lambda_S(d\alpha) f(\alpha)\exp\{-U_S(\alpha, x_S^*)\} \tag{4.2}$$

is continuous on $\mathbf{X}$. This follows from the assumed boundedness conditions on the interaction potential $V$ and from well-known continuity properties of translation operators acting on integrable functions on Euclidean space.

We say that a probability $\mu$ on $\mathbf{X}$ is Gibbsian over $S$ if

$$\begin{aligned}&\int \mu(dx) f(x_S) g(x_S^*)\\ &\quad = \int_{\mathbf{X}_S^*} \mu_S^*(d\beta) g(\beta)\{Z_S(\beta)\}^{-1} \int_{\mathbf{X}_S} \lambda_S(d\alpha) f(\alpha)\exp\{-U_S(\alpha,\beta)\}\end{aligned}$$

for $f, g$ bounded and measurable on $\mathbf{X}_S$ and $\mathbf{X}_S^*$ where $\mu_S^*$ is a fixed probability on $\mathbf{X}_S^*$. From continuity of the functions (4.2) it follows that the collection of all such probabilities on $\mathbf{X}$ is vaguely compact. It is easy to check that if $\mu$ is Gibbsian over one $S$ then it is Gibbsian over all smaller $S$ and from this follows the existence of at least one probability $\mu$ which is Gibbsian over all $S$. We call any such $\mu$ a Gibbs probability. Now we are ready for

**Lemma 4.1.** *Let $\mu$ be a Gibbs probability on $\mathbf{X}$. Then for $f(x,y)$ nonnegative and jointly Borel measurable on $\mathbf{X} \times \mathbf{X}$*

$$\begin{aligned}&\int \mu(dx) \int \pi(x,dy) f(x,y)\\ &\quad = \rho \int di \int \pi^\circ(i,dj) \int \mu(dx) I[x+\varepsilon_i, x+\varepsilon_j \text{ are in } \mathbf{X}] f(x+\varepsilon_i, x+\varepsilon_j).\end{aligned}$$

*In particular $\mu$ is symmetric.*

**Proof.** Clearly

$$\int \mu(dx) \int \pi(x,dy) f(x,y) = \int \mu(dx) \sum_i x(i)\exp\Big\{\sum_k x(k)V(i-k)\Big\}(x,i)$$

with $g$ defined on $\mathbf{X} \times R^d$ by

$$g(x,i) = \int \pi^\circ(i,dj) I(x - \varepsilon_i + \varepsilon_j \text{ is in } X) f(x, x + \varepsilon_j - \varepsilon_i).$$

The lemma will be proved if we establish

$$\begin{aligned} &\int \mu(dx) \sum_i x(i) \exp\left\{\sum_k x(k) V(i-k)\right\} g(x,i) \\ &\qquad = \rho \int di \int \mu(dx) I(x + \varepsilon_i \text{ is in X}) g(x + \varepsilon_i, i). \end{aligned} \tag{4.3}$$

If suffices to establish (4.3) instead for $h(x,i) = \exp\{-\mathfrak{S}\varphi(x)\}\psi(i)$ with $\varphi, \psi \geq 0$ on $R^d$ and supported by a bounded open set $S$. The left side of (4.3) with $g$ replaced by $h$

$$\begin{aligned} &= \int \mu(dx) \sum_i x(i)\psi(i) \exp\{-\mathfrak{S}\varphi(x)\} \exp\left\{\sum_k x(k)\, V(i-k)\right\} \\ &= \int \mu_S^*(d\beta)\{Z_S(\beta)\}^{-1} \int \lambda_S(d\alpha) \sum_i \alpha(i)\psi(i) \exp\{-\mathfrak{S}\varphi(\alpha)\} \\ &\quad \cdot \exp\left\{\sum_j \alpha(j) V(i-j) + \sum_k \beta(k) V(i-k)\right\} \exp\{-U_S(\alpha,\beta)\}. \end{aligned}$$

But it is well known that

$$\begin{aligned} &\int \lambda_S(d\alpha) \sum_i \alpha(i) h^\#(\alpha, i) \\ &\qquad = \rho \int_S di \int \lambda_S(d\alpha) h^\#(\alpha + \varepsilon_i, i) \end{aligned} \tag{4.4}$$

whenever $h^\# \geq 0$ is Borel measurable on $\mathbf{X}_S \times S$. In verifying (4.4) it suffices to consider the special case

$$h^\#(\alpha,i) = \exp\{-\mathfrak{S}\varphi^+(\alpha)\}\psi^+(i)$$

where $\psi^+ = 1_{A_1}$; $\varphi^+ = \sum_{i=1}^N c_i 1_{A_i}$ with $c_1, \ldots, c_N > 0$ and $A_1, \ldots, A_N$ disjoint Borel subsets of cl$(S)$. Then

$$\begin{aligned} &\int \lambda_S(d\alpha) \sum_i \alpha(i) h^\#(\alpha,i) \\ &\qquad = \sum_{k=0}^{\infty} \exp(-\rho|A_1|)\{\rho^k |A_1|^k/k!\} k e^{-kc_1} \\ &\qquad\quad \cdot \prod_{j=2}^{N} \exp\{-\rho|A_j| + \rho|A_j|\exp(-c_j)\} \\ &\qquad = \rho|A_1|\exp(-c_1) \int \lambda_S(d\alpha) \exp\{-\mathfrak{S}\varphi^+(\alpha)\} \\ &\qquad = \rho \int_S di \int \lambda_S(d\alpha) h^\#(\alpha + \varepsilon_i, i) \end{aligned}$$

and (4.4) follows. We apply (4.4) with

$$h^*(\alpha,i) = \psi(i)\exp\{-\mathbb{S}\varphi(\alpha)\}$$
$$\cdot \exp\Big\{\sum_j \alpha(j)V(i-j) + \sum_k \beta(k)V(i-k)\Big\}\exp\{-U_S(\alpha,\beta)\}.$$

Then

$$h^{\#}(\alpha+\varepsilon_i,i) = \psi(i)e^{-\varphi(i)}\exp\{-\mathbb{S}\varphi(\alpha)\}\exp\{-U_S(\alpha,\beta)\}I[U_S(\alpha+\varepsilon_i,\beta) < +\infty]$$

and so the left side of (4.3) with $h$ instead of $g$

$$= \int \mu_S^*(d\beta)\{Z_S(\beta)\}^{-1}\rho\int di\int \lambda_S(d\alpha)\psi(i)e^{-\varphi(i)}$$
$$\cdot \exp\{-\mathbb{S}\varphi(\alpha)\}\exp\{-U_S(\alpha,\beta)\}I[U_S(\alpha+\varepsilon_i,\beta) < +\infty]$$
$$= \rho\int di\int \mu(dx)I[x+\varepsilon_i \text{ is in } \mathbf{X}]h(x+\varepsilon_i,i)$$

which proves (4.3) for $h$ and therefore for $g$ after a passage to the limit.

Finally we prove that side condition 1.5 is satisfied for any Gibbs probability $\mu$. For our subbasis we take open sets $D_1 = \{x\colon \mathbb{S}1_B(x) = 1\}$ with $B$ an open ball of radius less than $r$ and open sets $D_2 = \{x\colon \mathbb{S}1_{\mathrm{cl}(G)} = 0\}$ with $G$ a bounded open subset of $R^d$ having a smooth boundary. To handle $D_1$ let $\varphi_n$ be as in Lemma 2.1(i). Clearly $f_n = 1 - \exp\{S\log(1-\varphi_n)\}$ is continuous and supported by $D_1$ and $f_n \to 1_{D_1}$ [a.e. $\mu$]. Moreover by Lemma 4.1

$$\int \mu(dx)\int \pi(x,dy)|f_n(x)-f_n(y)|^2 \le \rho\int di\int \pi^\circ(i,dj)|\varphi_n(i)-\varphi_n(j)|^2$$

and it follows that $1_{D_1}$ is in $\mathbf{F}^{D_1}$. To obtain this result for $D_2$ take $\varphi_n$ as in Lemma 2.1(ii) and define $f_n = \exp\{\mathbb{S}\log(1-\varphi_n)\}$.

**Appendix. Proof of Theorem 1.3 under the side condition 1.6.** We begin by proving a general result which may be of independent interest. Let $\mathbf{X}$ and $\mu$ be as in the text and let $(\mathbf{F}^*, E^*)$ be a Dirichlet space relative to $L^2(\mathbf{X},\mu)$ and satisfying the following regularity condition.

A.1. $\mathbf{F}^* \cap C_{\mathrm{com}}(\mathbf{X})$ is uniformly dense in $C(\mathbf{X})$.

Let $q(x) \ge 0$ be locally bounded on $\mathbf{X}$ and define

$$\mathbf{F}^q = \Big\{f \text{ in } \mathbf{F}\colon \int dx\, q(x)f^2(x) < +\infty\Big\},$$

$$E^q(f,g) = E^*(f,g) + \int dx\, q(x)f(x)g(x).$$

The pair $(\mathbf{F}^q, E^q)$ is easily seen to be a Dirichlet space relative to $L^2(\mathbf{X},\mu)$. Our general result effectively identifies it as the Dirichlet space derived from $(\mathbf{F}^*, E^*)$ by "killing at the rate $q$."

**Theorem A.1.** *Let* $\{G_u^*, u > 0\}$ *be the resolvent corresponding to the given Dirichlet space* $(\mathbf{F}^*, E^*)$ *and let* $\{G_u^q, u > 0\}$ *correspond to the Dirichlet space*

$(\mathbf{F}^q, E^q)$. *Then for* $u > 0$

$$G_u = G_u^q + G_u^q q G_u = G_u^q + G_u q G_u^q.$$

**Proof.** If $(\mathbf{F}^*, E^*)$ is regular so that the construction in [13, §2] is applicable we define $G_u^{q\#} f(x) = \mathcal{E}_x^* \int_0^{\sigma(u)} dt\, f(X_t)$ where

$$\sigma(u) = \inf\left\{t > 0 : ut + \int_0^t ds\, q(X_s) > R\right\}$$

with $R$ the usual random time independent of the trajectory variables and exponentially distributed with density $e^{-l} I$ $(l \geq 0)$. A simple computation establishes

$$G_u^* = G_u^{q\#} + G_u^* q G_u^{q\#} = G_u^{q\#} + G_u^{q\#} q G_u^*. \tag{A.1}$$

If $(\mathbf{F}^*, E^*)$ is not regular we use the appendix in [13] together with the above construction to establish the existence of a resolvent $\{G_u^{q\#}, u > 0\}$ satisfying (A.1). Denote the corresponding Dirichlet space by $(\mathbf{F}^{q\#}, E^{q\#})$. The theorem will be proved if we show that actually $(\mathbf{F}^{q\#}, E^{q\#}) = (\mathbf{F}^q, E^q)$. We begin by establishing

$$\text{(A.2)} \qquad \begin{aligned} &\mathbf{F}^{q\#} \subset \mathbf{F}^q, \\ &E^{q\#}(f,f) \geq E^q(f,f), \qquad f \text{ in } \mathbf{F}^{q\#}, f \geq 0. \end{aligned}$$

We remark first that if $\{G_u^1, u > 0\}$ and $\{G_u^2, u > 0\}$ are resolvents with corresponding Dirichlet spaces $(\mathbf{F}^1, E^1)$ and $(\mathbf{F}^2, E^2)$ and if $(G_u^1 f, f) \leq (G_u^2 f, f)$ for $f \geq 0$ in $L^2(\mathbf{X}, \mu)$, then $\mathbf{F}^1 \subset \mathbf{F}^2$ and also $E^1(f,f) \geq \mathrm{E}^2(f,f)$ for $f \geq 0$ in $\mathbf{F}^1$. This follows from the fundamental relation

$$E^i(f,f) = \lim u \int \mu(dx)\{f - uG_u^i f\}(x) f(x) \qquad (u \uparrow \infty)$$

which is an easy consequence of the spectral theorem. Thus (A.1) implies that $\mathbf{F}^{q\#} \subset \mathbf{F}^*$ and so for $g \geq 0$ in $L^2(\mathbf{X}, \mu)$

$$\begin{aligned} &E_u^*(G_u^{q\#} g, G_u^{q\#} g) \\ &\quad = \operatorname{Lim} \int \mu(dx) v \{G_u^{q\#} g - v G_{u+v}^* G_u^{q\#} g\}(x) G_u^{q\#} g(x) \\ &\quad = \operatorname{Lim}\left\{\int \mu(dx) v \{G_u^* g - G_{u+v}^* G_u^* g\}(x) G_u^{q\#} g(x)\right. \\ &\qquad\qquad \left. - \int \mu(dx) v \{G_u^* q G_u^{q\#} g - v G_{u+v}^* G_u^* q G_u^{q\#} g\}(x) G_u^{q\#} g(x)\right\} \\ &\quad = \operatorname{Lim}\left\{\int \mu(dx) v G_{u+v}^* g(x) G_u^{q\#} g(x)\right. \\ &\qquad\qquad \left. - \int \mu(dx) v G_{u+v}^* q G_u^{q\#} g(x) G_u^{q\#} g(x)\right\} \end{aligned}$$

with all limits taken as $v \uparrow \infty$, and by Fatou's lemma

$$E_u^*(G_u^{q\#} g, G_u^{q\#} g) \leq E_u^{q\#}(G_u^{q\#} g, G_u^{q\#} g) - \int \mu(dx)\, q(x) \{G_u^{q\#}(x)\}^2$$

which gives (A.2) after dropping the subscript $u$ from $E_u^*$ and $E_u^{q\#}$, approximating $f \geq 0$ in $\mathbf{F}^{q\#}$ by $uG_u^{q\#} f$ and once more applying Fatou's lemma. By the remark following (A.2), the theorem will be completely proved if we show that

$$\int \mu(dx) G_u^q f(x) f(x) \leq \int \mu(dx) G_u^{q\#} f(x) f(x) \tag{A.3}$$

for $f \geq 0$ in $L^2(\mathbf{X}, dx)$. It suffices to show this when $G_u^{q\#}$ is replaced by $G_u^{q'\#}$ defined as above except that $q$ is replaced by a bounded nonnegative function $q'$ dominated by $q$. It is easy to check that $\mathbf{F}_u^{q'\#} = \mathbf{F}^*$ and that

$$E^{q'\#}(f,g) = E^*(f,g) + \int \mu(dx) q'(x) f(x) g(x)$$

for $f$, $g$ in $\mathbf{F}^*$. Then for $\varphi \geq 0$ in $\mathbf{F}^* \cap C_{\text{com}}(\mathbf{X})$ and therefore in $\mathbf{F}^q$,

$$\int \mu(dx) f(x) \varphi(x) = E_u^q(G_u^q f, \varphi) \geq E_u^{q'\#}(G_u^q f, \varphi).$$

If $(\mathbf{F}^*, E^*)$ (and therefore $(\mathbf{F}^{q'\#}, E^{q'\#})$) were regular we could approximate $G_u^{q'\#} f$ by such $\varphi$ and (A.3) would follow. To handle the general case let $\mathbf{F}^{q'\#\#}$ be the $E_1^{q'\#}$ closure of $\mathbf{F}^* \cap C_{\text{com}}(\mathbf{X})$ and denote by $\{G_u^{q'\#\#}, u > 0\}$ the resolvent which corresponds to the Dirichlet space $(\mathbf{F}^{q'\#\#}, E^{q'\#})$. Then $G_u^{q'\#\#} f$ can be approximated by such $\varphi$ and so (A.3) will follow if we establish

$$\int \mu(dx) G_u^{q'\#\#} f(x) f(x) \leq \int \mu(dx) G_u^{q'\#} f(x) f(x). \tag{A.4}$$

But for $f$, $g$ in $L^2(\mathbf{X}, \mu)$

$$E_u^{q'\#}(G_u^{q'\#} f - G_u^{q'\#\#} f, G_u^{q\#} g) = 0$$

and so $G_u^{q'\#\#} f$ is the orthogonal projection of $G_u^{q'\#} f$ onto $\mathbf{F}^{q'\#\#}$. Then

$$E_u^{q'\#}(G_u^{q'\#\#} f, G_u^{q'\#\#} f) \leq E_u^{q'\#}(G_u^{q'\#} f, G_u^{q'} f)$$

which is equivalent to (A.4).

We show now that Theorem 1.3 follows from Theorem A.1 under the side condition 1.6. Consider an open subset $D$ of $\mathbf{X}$ satisfying (1.1) and denote its closure by $D^*$. The killed resolvent $\{G_u^{D^*}, u > 0\}$ is defined just as for open $D$ and the arguments of [13, §3] can be modified in an obvious manner to show that the $G_u^{D^*}$ are symmetric and that the corresponding Dirichlet space $(\mathbf{F}^{D^*}, \mathbf{E}^{D^*})$ is given by $\mathbf{F}^{D^*} = \{f \text{ in } \mathbf{F}: f = 0 \text{ q.e. on } \mathbf{X} - D^*\}$, $E^{D^*}(f,g) = E(f,g)$, $f$, $g$ in $\mathbf{F}^{D^*}$. Suppose now that $f$ in $L^2(\mathbf{X}, \mu)$ has a quasi-continuous version $f^*$ belonging to $\mathbf{F}$ and that $f = 0$ [a.e. $\mu$] on $\mathbf{X} - D^*$. Since $\mathbf{X} - D^*$ is open, Lemma 1.15 in [13] implies that also $f^* = 0$ q.e. on $\mathbf{X} - D^*$ and therefore $f^*$ belongs to $\mathbf{F}^{D^*}$. (It is crucial for this argument that $X - D^*$ be open.) It follows that modulo quasi-continuous refinements

$$\mathbf{F}^{D^*} = \{f \text{ in } \mathbf{F}: f = 0 \text{ [a.e. } dx] \text{ on } \mathbf{X} - D^*\},$$
$$E^{D^*}(f,g) = \tfrac{1}{2}\pi^{D^*}\langle f,g\rangle + \int_{D^*} \mu(dx)\pi(x,\mathbf{X} - D^*)f(x)g(x).$$

Consider the pair

$$\mathbf{F}^* = \{f \text{ in } L^2(D,dx)\}: \pi^{D^*}\langle f,f\rangle \text{ is finite},$$
$$E^*(f,g) = \tfrac{1}{2}\pi^{D^*}\langle f,g\rangle.$$

It is easy to check that $(\mathbf{F}^*, E^*)$ is a conservative Dirichlet space relative to $L^2(D,\mu)$ (that is, $uG_u^* = 1$ [a.e. $\mu$] on $D$ for $u > 0$) and that (again modulo quasi-continuous refinements)

$$\mathbf{F}^{D^*} = \left\{f \text{ in } \mathbf{F}^*: \int_D \mu(dx)\,\pi(x,\mathbf{X} - D^*)\{f(x)\}^2 \text{ is finite}\right\},$$
$$E^{D^*}(f,g) = E^*(f,g) + \int_D dx\,\pi(x,\mathbf{X} - D^*)f(x)g(x).$$

The regularity condition A.1 for $(\mathbf{F}^*, E^*)$ follows directly from the regularity condition 1.3 for $(\mathbf{F}, E)$. Thus Theorem A.1 is applicable with $q(x) = \pi(x,\mathbf{X} - D^*)$ and

$$\text{(A.5)} \qquad \begin{aligned} 1 = uG_u^* 1 &= G_u^{D^*}\{u1 + \pi(\cdot,X - D^*)uG_u^* 1\} \\ &= G_u^{D^*}\{u1 + \pi(\cdot,\mathbf{X} - D^*)\} \end{aligned}$$

[a.e. $\mu$] on $D$ and it follows in particular that

$$1 = G_u^D\{u1 + \pi(\cdot,\mathbf{X} - D^*)\}$$

and therefore

$$\text{(A.6)} \qquad 1 \le G^D\{u1 + \pi(\cdot,\mathbf{X} - D)\}$$

[a.e. $\mu$] on $D$. The opposite inequality is elementary. If $q(x)$ is any bounded nonnegative function on $D$ dominated by $\pi(\cdot,\mathbf{X} - D)$, then

$$E^q(f,g) = E^D(f,g) - \int \mu(dx)q(x)f(x)g(x)$$

is easily seen to be a Dirichlet norm on $\mathbf{F}^D$ and by Theorem A.1 (for a special case that is easily verified directly)

$$1 \ge uG_u^q 1 = G_u^D\{u1 + q\}.$$

This establishes (1.2) up to $\mu$-null sets on $D$. Identity quasi-everywhere on $D$ follows since both sides can be approximated from below quasi-everywhere on $D$ by applying $vG_{u+v}^D$ and letting $v \uparrow \infty$.

## ERRATUM
## Added in proof April 2, 1974

Recent work by the author makes it clear that the side conditions 1.5 and 1.6 are irrelevant for the identification of the measure $\pi(x, dy)$ as "Levy measures." (See Chapter 2 in the author's monograph *Symmetric Markov Processes* which is now in preparation.) However the real point of this erratum is to point out that Theorem A. 1 in the appendix is false without an additional hypothesis such as regularity for $(\mathbf{F}^*, E^*)$. This is in fact a condition on $q(x)$. For a counterexample to the theorem as stated let $\mathbf{X} = (0, \infty)$, let $\mu$ be Lebesgue measure, let $(\mathbf{F}^*, E^*)$ be the Dirichlet space which corresponds to reflecting Brownian motion and let $q(x)$ be such that $q(x)$ is not integrable near $0$ but $xq(x)$ is.

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